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Periodic Homogenization for Hypoelliptic Diffusions

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We study the long time behavior of an Ornstein–Uhlenbeck process under the influence of a periodic drift. We prove that, under the standard diffusive rescaling, the law of the particle position converges weakly to the law of a Brownian motion whose covariance can be expressed in terms of the solution of a Poisson equation. We also derive upper bounds on the convergence rate in several metrics.

KEY WORDS: periodic homogenization; hypoellipticity; martingale central limit theorem; convergence rate; Wasserstein metric.

1. INTRODUCTION

In this paper we study the long time behavior of solutions of the following Langevin equation:

$$\tau \ddot{x}(t) = v(x(t)) - \dot{x}(t) + \sigma \dot{\beta}(t), \quad x(t) \in \mathbb{R}^n, \tag{1.1}$$

where $\beta(t)$ is a standard Brownian motion and σ, $\tau > 0$. The parameter τ can be thought of as a nondimensional particle relaxation time, which measures the inertia of the particle. The drift term v is taken to be smooth, periodic with period 1 in all directions; further, it is assumed that it satisfies an appropriate centering condition.

It is well known that as τ tends to 0, and provided that $v(x)$ is Lipschitz continuous, the solution of (1.1) converges with probability 1 to the solution of the Smoluchowski equation

$$\dot{z}(t) = v(z(t)) + \sigma \dot{\beta}(t), \quad x(t) \in \mathbb{R}^n, \tag{1.2}$$

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uniformly over every finite time interval, see e.g. ref. 17. The problem of homogenization for equation (1.2) has been studied extensively over the last three decades for periodic[2,4,20] as well as random[5,13,15] drifts. For the case where $v(z)$ is a smooth, periodic field which is centered with respect to the invariant measure of the process, it is not hard to prove[4] that the rescaled process $\varepsilon z(t/\varepsilon^2)$ converges, as ε tends to 0, to a Brownian motion with a positive definite covariance matrix $\mathcal{K}$. The proof of this functional central limit theorem is based on the proof of a spectral gap for the generator of the process $z(t)$.

The long time behavior of particles with non–negligible inertia, whose evolution is governed by equation (1.1) has been investigated by Freidlin and coworkers in a series of papers[8–11]. Among other things, Hamiltonian systems under weak deterministic and random perturbations were studied in these papers:

$$\tau \ddot{x} = -\nabla V(x) + \varepsilon(-\kappa \dot{x} + \gamma) + \sqrt{\varepsilon}\sigma \dot{\beta}, \tag{1.3}$$

with $\kappa,\ \gamma \in \mathbb{R}$. It was shown that, under appropriate assumptions on the potential $V(x)$, the rescaled process $\{x(t/\varepsilon),\ y(t/\varepsilon)\}$ converges weakly to a diffusion process on a graph corresponding to the Hamiltonian of the system $H = \frac{1}{2}\tau \dot{x}^2 + V(x)$.

On the other hand, the problem of homogenization for (1.1) has been investigated less. This is not surprising since the hypoellipticity of the generator of the process (1.1) renders the derivation of the necessary spectral gap estimates more difficult. Homogenization results for the solution $x(t)$ of (1.1) have been obtained, to our knowledge, only for the case where the drift $v(x)$ is the gradient of a potential. In this case the invariant measure of the process $\{x(t),\ \dot{x}(t)\}$ is explicitly known and this fact simplifies considerably the analysis. This problem was analyzed for periodic[23] as well as random potentials.[22] In both cases it was shown that the particle position converges, under the diffusive rescaling, to a Brownian motion with a positive covariance matrix $\mathcal{K}$. The proofs of these homogenization theorems are based on the techniques developed for the study of central limit theorems for additive functionals of Markov processes[14], together with a regularization procedure for appopriate degenerate Poisson equations. Related questions for subelliptic diffusions have also been investigated.[18,19,3]

The purpose of this paper is to prove a central limit theorem for the solution of the Langevin equation (1.1) with a general periodic smooth drift $v(x)$ and, further, to obtain bounds on the convergence rate. The proof of our homogenization theorem relies on the strong ergodic properties of hypoelliptic diffusions. The techniques developed in refs. 6, 7 enable us to prove the existence of a unique, smooth invariant measure for (1.1) and to obtain precise estimates on the solution of the Poisson equation $-\mathcal{L}f = g$, where $\mathcal{L}$ is the generator of the process (1.1) and the function g is smooth and centered with respect to the invariant measure. Based on these estimates it is rather straightforward to show that the rescaled particle position $\varepsilon x(t/\varepsilon^2)$ converges to a Brownian motion, using the techniques developed in ref. 14. Obtaining bounds on the rate of convergence requires more work. For this, we need to identify the limiting Brownian motion and to introduce an additional Poisson equation. Furthermore, we need to control the 1-Wasserstein distance between two probability measures by the distance between their characteristic functions. This is accomplished using ideas from refs. 12, 25.

The sequel of this paper is organized as follows. In section 2 we introduce the notation that we will be using throughout the paper and we present our main result, Theorem 2.1. In section 3 we prove various estimates on the invariant measure of (1.1) and the solution of the cell problem, and we also derive estimates on moments of the particle velocity. The proof of Theorem 2.1 is presented in section 4. Finally, section 5 is reserved for a few concluding remarks.

2. NOTATION AND RESULTS

Consider the following Langevin equation in $\mathbb{R}^n$:

$$\tau \ddot{x}(t) = v(x(t)) - \dot{x}(t) + \sigma \dot{\beta}(t)\,, \tag{2.1}$$

with initial conditions $x(0) = x$, $\dot{x}(0) = (\sqrt{\tau})^{-1} y$. We assume throughout this paper that $v \in \mathcal{C}^\infty(\mathbb{T}^n)$. Introducing $y(t) = \sqrt{\tau}\dot{x}(t)$, we rewrite (2.1) as a first order stochastic differential equation:

$$\begin{aligned} dx(t) &= \frac{1}{\sqrt{\tau}} y(t)\, dt, \\ dy(t) &= \frac{1}{\sqrt{\tau}} v(x(t))\, dt - \frac{1}{\tau} y(t)\, dt + \frac{\sigma}{\sqrt{\tau}} d\beta(t)\,. \end{aligned} \tag{2.2}$$

We denote by $\mathcal{L}$ the generator of the process $\{x(t), y(t)\}$:

$$\mathcal{L} = \frac{1}{\sqrt{\tau}} \left(y \cdot \nabla_x + v(x) \cdot \nabla_y\right) + \frac{1}{\tau} \left(-y \cdot \nabla_y + \frac{\sigma^2}{2} \Delta_y\right). \tag{2.3}$$

By Theorem 3.1 below, the process $\{x(t), y(t)\}$ admits a unique, smooth invariant measure, denoted by $\mu(dx, dy)$.

Consider now the Poisson equation

$$-\mathcal{L}\Phi = \frac{1}{\sqrt{\tau}}\, y\,. \tag{2.4}$$

This equation is posed on $\mathbb{T}^n \times \mathbb{R}^n$. In accordance to the terminology of periodic homogenization, we will be referring to equation (2.4) as the *cell problem*, even though its solutions are periodic only with respect to x. This equation has a unique, smooth solution in the appropriate function space by Theorem 3.3, provided that $\int v(x)\, \nu(dx, dy) = 0$. We define the symmetric, positive $n \times n$ matrix $\mathcal{K}$ such that

$$\mathcal{K}^2 = \frac{\sigma^2}{\tau} \int \nabla_y \Phi \otimes \nabla_y \Phi \, d\mu\,. \tag{2.5}$$

The main result of this paper is that the particle position, under the standard diffusive rescaling, converges weakly to a Brownian motion with covariance $\mathcal{K}^2$. We furthermore give upper bounds on the rate of convergence in the following metrics.

Let $\mathcal{B}$ denote a separable Banach space and $\mathcal{B}^*$ be its dual space. Given two measures μ_1 and μ_2 on $\mathcal{B}$, we also denote by $\mathcal{C}(\mu_1, \mu_2)$ the set of all measures on $\mathcal{B}^2$ with marginals μ_1 and μ_2. With these notations, we define the following metric on the space of probability measures on $\mathcal{B}$ with finite p-moment:

$$|||\mu_1 - \mu_2|||_p^p = \sup_{\ell \in \mathcal{B}^*} \inf_{\mu \in \mathcal{C}(\mu_1,\mu_2)} \int_{\mathcal{B}^2} \frac{|\ell(x) - \ell(y)|^p}{\|\ell\|^p}\, \mu(dx, dy)\,. \tag{2.6}$$

This distance is close in spirit to the p-Wasserstein distance

$$|||\mu_1 - \mu_2|||_{p,W}^p = \inf_{\mu \in \mathcal{C}(\mu_1,\mu_2)} \int_{\mathcal{B}^2} \|x - y\|^p\, \mu(dx, dy)\,,$$

so we will refer to it as the weak p-Wasserstein distance. Note that the distance (2.6) gives a locally uniform bound on the distance between characteristic functions $\chi_\mu(\ell) = \int e^{i\ell(x)}\, \mu(dx)$:

$$|\chi_{\mu_1}(\ell) - \chi_{\mu_2}(\ell)| \leqslant \|\ell\|\; |||\mu_1 - \mu_2|||_p\,. \tag{2.7}$$

In particular one has $|||\mu_1 - \mu_2|||_p = 0$ if and only if $\mu_1 = \mu_2$.

In order to simplify notations, we define the fast processes $y_t^\varepsilon = y(\varepsilon^{-2}t)$ and $x_t^\varepsilon = x(\varepsilon^{-2}t)$. We will also from now on use the notation

$\mathcal{B} = \mathcal{C}([0,T], \mathbb{R}^n)$, for a value $T > 0$ that remains fixed throughout this paper. Moreover, we define by $\pi_k : \mathcal{B} \to \mathcal{C}([0,T], \mathbb{R})$ the projection given by $(\pi_k x)(t) = \langle k, x(t)\rangle$. Given a measure μ on a space $\mathcal{M}$ and a measurable function $f : \mathcal{M} \to \mathcal{N}$, we denote by $f^*\mu$ the measure on $\mathcal{N}$ given by $(f^*\mu)(A) = \mu(f^{-1}(A))$.

Now we are ready to state the homogenization theorem.

Theorem 2.1. Let $x(t)$ be the solution of (2.1), in which the velocity field $v \in \mathcal{C}^\infty(\mathbb{T}^n)$ satisfies $\int v(x)\, \nu(dx, dy) = 0$. For $T > 0$ fixed, denote by μ_ε the measure on $\mathcal{B}$ given by the law of the rescaled process $\varepsilon x_t^\varepsilon$ and by μ the law of a Brownian motion on $\mathbb{R}^n$ with covariance $\mathcal{K}^2$ as defined in (2.5). Then μ_ε converges weakly to μ and one has the following bounds on the convergence rate.

- For every $p \geqslant 1$ and $\alpha \in \left(0, \frac{1}{2}\right)$, there exists a constant C such that

$$|||\mu_\varepsilon - \mu|||_p \leqslant C\varepsilon^\alpha\,, \tag{2.8}$$

  for every $\varepsilon \in (0,1)$.
- For every $p \geqslant 1$ and $\alpha \in \left(0, \frac{1}{2}\right)$, there exists a constant C such that

$$\left|\left|\left|\pi_k^*\mu_\varepsilon - \pi_k^*\mu\right|\right|\right|_{p,W} \leqslant C\varepsilon^\alpha\,, \tag{2.9}$$

  for every $k \in \mathbb{R}^n$ with $\|k\| \leqslant 1$ and every $\varepsilon \in (0,1)$.
- For every $\beta < \frac{1}{20(n+3)^2}$, there exists a constant C such that

$$|||\mu_\varepsilon - \mu|||_{1,W} \leqslant C\varepsilon^\beta\,, \tag{2.10}$$

  for every $\varepsilon \in (0,1)$.

Remark 2.2. The condition $\int v(x)\, \nu(dx, dy) = 0$ ensures that there is no ballistic motion involved. In the general case, one can write $\bar{v} = \int v(x)\, \nu(dx, dy)$ and define $\varepsilon x_t^\varepsilon = \varepsilon x(\varepsilon^{-2} t) - \varepsilon^{-1}\bar{v}t$. Then, Theorem 2.1 holds for $\varepsilon x_t^\varepsilon$.

Remark 2.3. If $n = 1$, the bound (2.9) is much stronger than the bounds (2.8) and (2.10). If $n > 1$ however, this bound does not imply any form of convergence $\mu_\varepsilon \Rightarrow \mu$. It is indeed possible to construct two Gaussian stochastic processes $x(t)$ and $y(t)$ with values in $\mathbb{R}^2$ such that the law of x differs from the law of y and such that, for every $k \in \mathbb{R}^2$, the law of $\langle k, x\rangle$ is identical to the law of $\langle k, y\rangle$. As an example, choose three i.i.d. Gaussian centered random variables a_1, a_2, a_3 and define

$$x_1(t_1)=a_1 \qquad x_2(t_1)=a_2 \qquad x_1(t_2)=a_3 \qquad x_2(t_2)=a_1$$
$$y_1(t_1)=a_1 \qquad y_2(t_1)=a_2 \qquad y_1(t_2)=a_2 \qquad y_2(t_2)=a_3\,.$$

It is an easy exercise to check that these two processes possess the required properties.

Remark 2.4. Convergence in the weak p-Wasserstein distance alone does *not* imply weak convergence, as the space of probability measures on $\mathcal{B}$ is not complete under $|\!|\!|\cdot|\!|\!|_p$. This can be seen by taking $\mathcal{B}=\ell^2$ and choosing for μ_n the Gaussian measure with covariance

$$Q_n=\mathrm{diag}\big(1,\tfrac{1}{2},\ldots,\tfrac{1}{n},0,\ldots\big)\,.$$

It is straightforward to check that this forms a Cauchy sequence with respect to $|\!|\!|\cdot|\!|\!|_p$, but does not converge to any measure supported in ℓ^2. (It does however converge weakly to a limiting measure in a weaker topology, and this is always the case.) In our situation, the additional control we have on the regularity of the processes involved allows to overcome this problem.

Remark 2.5. The covariance, or *effective diffusivity*, $\mathcal{K}^2$ of the limiting Brownian motion depends on the σ and τ. It is shown in ref. 21 that as τ tends to 0 the covariance $\mathcal{K}^2$ converges to the one obtained from the homogenization of equation 1.2. We refer to ref. 21 for further properties of the effective diffusivity, together with numerical experiments for various fields $v(x)$.

Remark 2.6. For simplicity, we choose the molecular diffusion σ to be a constant scalar. Taking for σ a positive definite matrix would only require a slight change in our notations. We could even allow σ to depend on x in a smooth way, as long as it remains strictly positive definite for all $x\in\mathbb{T}^n$. The results from refs. 6, 7 then still apply and one can check that all the bounds obtained in section 3 still hold. Since the proof of Theorem 2.1 itself never uses the fact that σ is constant, all of our results immediately carry over to this case.

Remark 2.7. For simplicity, we assumed the initial condition (x,y) to be deterministic. However, it is easy to check that all our arguments work for randomly distributed initial conditions provided that they are independent of the driving noise and that $\mathbf{E}\exp\frac{1}{2}\|\sigma^{-1}y\|^2<\infty$.

Remark 2.8. One may wonder if it is possible to show convergence of μ_ε to μ in a stronger topology, like the one given by the total variation distance. Since the sample paths of the Brownian motion are almost surely not differentiable, whereas $t \mapsto \varepsilon x_t^\varepsilon$ almost surely is, the measures μ_ε and μ are actually mutually singular for every $\varepsilon > 0$. Concerning the distributions for a fixed time t, one expects from a formal expansion that the density of the law of $\varepsilon x_t^\varepsilon$ is given by $\mu_\varepsilon^t(x) = \mu^t(x)\rho(\varepsilon^{-1}x) + \mathcal{O}(\varepsilon)$, where ρ is the periodic continuation of the density of the marginal (on the first component) of the invariant measure for the diffusion (2.2). It is straightforward to check that, unless ρ is constant, the total variation distance (*i.e.* the L^1 distance between densities in this case) between $\mu^t(x)\rho(\varepsilon^{-1}x)$ and $\mu^t(x)$ does not converge to 0 as $\varepsilon \to 0$.

The proof of Theorem 2.1 will be presented in section 4.

3. PRELIMINARY ESTIMATES

In this section we collect various estimates which are necessary for the proof of the homogenization theorem. In section 3.1 we study the structure of the invariant measure ν for (2.1). We show that it possesses a smooth density with respect to the Lebesgue measure and we derive sharp bounds for it. Further, we investigate the solvability of the Poisson equation

$$-\mathcal{L}f = h,$$

where h is a smooth function of x and y which is centered with respect to ν. We prove that equation (3.1) has a smooth solution which is unique in the class of functions which do not grow too fast at infinity.

In section 3.2 we derive estimates on exponential moments of the particle velocity. Roughly speaking, these estimates imply that the particle velocity grows very slowly with time.

3.1. Bounds on the Invariant Measure and on the Solution of the Poisson Equation

If $v = 0$, the invariant measure for (2.1) is given by $\nu = e^{-\frac{\|y\|^2}{\sigma^2}}\,dx\,dy$. This is "almost" true also in the case $v \neq 0$, as can be seen by the following result.

Theorem 3.1. Let ν be the invariant measure for (2.1) and denote by $\rho(x, y)$ its density with respect to the Lebesgue measure. Then, for every $\delta \in (0, 2\sigma^{-2})$ one can write

$$\rho(x, y) = e^{-\frac{\delta}{2}\|y\|^2} g(x, y)\,, \quad g \in \mathcal{S}, \tag{3.1}$$

where $\mathcal{S}$ denotes the Schwartz space of smooth functions with fast decay.

Proof. The proof follows the lines of refs. 6, 7. Denote by ϕ_t the (random) flow generated by the solutions to (2.1) and by $\mathcal{P}_t$ the semigroup defined on finite measures by

$$\big(\mathcal{P}_t\mu\big)(A)=\mathbf{E}\big(\mu\circ\phi_t^{-1}\big)(A)\,.$$

Since $\partial_t+\mathcal{L}$ is hypoelliptic, $\mathcal{P}_t$ maps every measure into a measure with a smooth density with respect to the Lebesgue measure. It can therefore be restricted to a positivity preserving contraction semigroup on $\mathrm{L}^1(\mathbb{T}^n\times\mathbb{R}^n,dx\,dy)$. The generator $\tilde{\mathcal{L}}$ of $\mathcal{P}_t$ is given by the formal adjoint of $\mathcal{L}$ defined in (2.3).

We now define an operator K on $\mathrm{L}^2(\mathbb{T}^n\times\mathbb{R}^n,dx\,dy)$ by closing the operator defined on $\mathcal{C}_0^\infty$ by

$$K=-e^{\frac{\delta}{2}\|y\|^2}\tilde{\mathcal{L}}e^{-\frac{\delta}{2}\|y\|^2}\,. \tag{3.2}$$

The operator K is then given by

$$\begin{aligned}K=&-\frac{\sigma^2}{2\tau}\Delta_y+\frac{\delta}{\tau}\Big(1-\frac{\delta\sigma^2}{2}\Big)\|y\|^2+\frac{1}{\tau}\big(\delta\sigma^2-1\big)\Big(y\cdot\nabla_y+\frac{n}{2}\Big)\\&+\frac{1}{\sqrt{\tau}}\big(y\cdot\nabla_x+v(x)\cdot\nabla_y\big)-\frac{n}{2\tau}\,.\end{aligned}$$

Note at this point that $\delta<2\sigma^{-2}$ is required to make the coefficient of $\|y\|^2$ in this expression strictly positive. This can be written in Hörmander's "sum of squares" form as

$$K=\sum_{i=1}^{2n}X_i^*X_i+X_0\,,$$

with

$$\begin{aligned}X_i&=\frac{\sigma}{\sqrt{2\tau}}\partial_{y_i} &&\text{if } i=1\ldots n,\\X_i&=\sqrt{\frac{\delta}{\tau}\Big(1-\frac{\delta\sigma^2}{2}\Big)}\,y_{i-n} &&\text{if } i=(n+1)\ldots 2n,\\X_0&=\frac{1}{\tau}\big(\delta\sigma^2-1\big)\Big(y\cdot\nabla_y+\frac{n}{2}\Big)+\frac{1}{\sqrt{\tau}}\big(y\cdot\nabla_x+v(x)\cdot\nabla_y\big)-\frac{n}{2\tau}\,.\end{aligned}$$

Since v is $\mathcal{C}^\infty$ on the torus, it can be checked in a very straightforward way that the assumptions of ref. 6, Theorem 5.5, are satisfied with $\Lambda^2 = 1 - \Delta_x - \Delta_y + \|y\|^2$. Combining this with ref. 6, Lemma 5.6, we see that there exists $\alpha > 0$ such that, for every $\gamma > 0$, there exists a positive constant C such that

$$\|\Lambda^{\alpha+\gamma} f\| \leqslant C\big(\|\Lambda^\gamma K f\| + \|\Lambda^\gamma f\|\big), \tag{3.3}$$

holds for every f in the Schwartz space. Looking at (3.3) with $\gamma = 0$, we see that K has compact resolvent. Since $e^{-\frac{\delta}{2}\|y\|^2}$ is an eigenfunction with eigenvalue 0 for K^*, it follows that K has also an eigenfunction with eigenvalue 0, let us call it g. It follows from (3.3) and a simple approximation argument that $\|\Lambda^\gamma g\| < \infty$ for every $\gamma > 0$, and therefore g belongs to the Schwartz space. Furthermore, an argument given for example in ref. 7, Prop 3.6 shows that g must be positive. Since one has furthermore

$$\tilde{\mathcal{L}} e^{-\frac{\delta}{2}\|y\|^2} g = 0,$$

the function ρ given by (3.1) is the density of the invariant measure of (2.1). This concludes the proof of Theorem 3.1. ∎

Before we give bounds on (2.4), we show the following little lemma.

Lemma 3.2. Let $\delta \in (0, 2\sigma^{-2})$ and let K be as in (3.2). Then, the kernel of K is one-dimensional.

Proof. Let $\tilde{g} \in \ker K$. Then, by the same arguments as above, $e^{-\frac{\delta}{2}\|y\|^2}\tilde{g}$ is the density of an invariant signed measure for $\mathcal{P}_t$. The ergodicity of $\mathcal{P}_t$ immediately implies $\tilde{g} \propto g$. ∎

Now we are ready to prove estimates on the solution of the Poisson equation (3.1).

Theorem 3.3. Let $h \in \mathcal{C}^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ with $D^\alpha_{x,y} h \in L^2(\mathbb{T}^n \times \mathbb{R}^n; e^{-\varepsilon\|y\|^2} dx dy)$ for every multiindex α and every $\varepsilon > 0$. Assume further that $\int h(x,y)\,\nu(dx\,dy) = 0$, where ν is the invariant measure for (2.1). Then, there exists a function f such that (3.1) holds. Moreover, for every $\delta > 0$, the function f satisfies

$$f(x,y) = e^{\frac{\delta}{2}\|y\|^2}\tilde{f}(x,y), \quad \tilde{f} \in \mathcal{S}. \tag{3.4}$$

Furthermore, for every $\delta \in (0, 2\sigma^{-2})$, f is unique (up to an additive constant) in $L^2(\mathbb{T}^n \times \mathbb{R}^n, e^{-\delta\|y\|^2} dx dy)$.

Proof. By hypoellipticity, if there exists a distribution f such that (2.4) holds, then f is actually a $\mathcal{C}^\infty$ function.

We start with the proof of existence. Fix $\delta \in (0, 2\sigma^{-2})$, consider the operator K^* which is the adjoint of the operator K defined in (3.2), and define the function

$$u(x, y) = h(x, y)\, e^{-\frac{\delta}{2}\|y\|^2}.$$

It is clear that if there exists $\tilde{f}$ such that $K^*\tilde{f} = u$, then $f = e^{\frac{\delta}{2}\|y\|^2}\tilde{f}$ is a solution to (3.1). Consider the operator K^*K. By the considerations in the proof of Theorem 3.1, K^*K has compact resolvent. Furthermore, the kernel of K^*K is equal to the kernel of K, which in turn by Lemma 3.2 is equal to the span of g. Define $\mathcal{H} = \langle g\rangle^\perp$ and define M to be the restriction of K^*K to $\mathcal{H}$. Since K^*K has compact resolvent, it has a spectral gap and so M is invertible. Furthermore, since $\mathcal{L}y = \tau^{-1/2}v(x) - \tau^{-1}y$, one checks easily that $f \in \mathcal{H}$, therefore $\tilde{f} = KM^{-1}u$ solves $K^*\tilde{f} = u$ and thus leads to a solution to (3.1).

Since K^* satisfies a similar bound to (3.3) and since $\|\Lambda^\gamma u\| < \infty$ for every $\gamma > 0$, the bound (3.4) follows as in Theorem 3.1. The uniqueness of u in the class of functions under consideration follows immediately from Lemma 3.2. ∎

Remark 3.4. Note that the solution f of (3.1) is probably not unique if we allow for functions that grow faster than $e^{\sigma^{-2}\|y\|^2}$.

Remark 3.5. The identity $y\tilde{\mathcal{L}}\rho = 0$, where $\tilde{\mathcal{L}}$ is the formal adjoint of $\mathcal{L}$, immediately yields that $\int y\, \nu(dx, dy) = \sqrt{\tau}\int v(x)\, \nu(dx, dy)$. In particular, the assumption that the drift is centered implies that y is also centered. Moreover, y clearly satisfies the smoothness and fast decay assumptions of Theorem 3.3. Hence, the theorem applies to each component of equation (2.4) and we can conclude that there exists a unique smooth vector valued function Φ which solves the cell problem and whose components satisfy estimate (3.4).

3.2. Estimates on the Particle Velocity

One has the following bound

Lemma 3.6. There exists a constant $\gamma > 0$ such that

$$\mathbf{E}\exp\Big(\frac{1}{2}\|\sigma^{-1}y(t)\|^2\Big) \leqslant \exp\Big(\frac{1}{2}\|\sigma^{-1}y\|^2 + \gamma t\Big),$$
$$\mathbf{E}\exp\Big(\frac{1}{8\tau}\int_0^t \|\sigma^{-1}y(s)\|^2\, ds\Big) \leqslant \exp\Big(\frac{1}{4}\|\sigma^{-1}y\|^2 + \frac{\gamma}{2}t\Big).$$

holds for any initial condition y and every $t > 0$.

Proof. Itôs formula yields immediately the existence of a constant γ such that

$$\frac{1}{2}\|\sigma^{-1}y(t)\|^2 \leqslant \frac{1}{2}\|\sigma^{-1}y\|^2 + \gamma t - \frac{1}{2\tau}\int_0^t \|\sigma^{-1}y(s)\|^2\,ds + \frac{1}{\sqrt{\tau}}\int_0^t \langle\sigma^{-1}y(s), d\beta(s)\rangle\,.$$

The first bound follows by exponentiating both sides and taking expectations. The second bound follows in a similar way after dividing both sides by 2. ∎

This yields the following:

Theorem 3.7. Let $\psi : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be such that

$$\sup_{x\in\mathbb{T}^n,\, y\in\mathbb{R}^n} \Big|\psi(x,y)\exp\Big(-\frac{1}{4}\|\sigma^{-1}y\|^2\Big)\Big| < \infty\,.$$

Then, there exist constants $C, \delta > 0$ such that

$$\mathbf{E}\big(\psi(x(t), y(t))\big) - \int_{\mathbb{T}^n\times\mathbb{R}^n} \psi(x,y)\,\nu(dx,dy) \leqslant C\exp\big(\|\sigma^{-1}y\|^2 - \delta t\big)\,. \tag{3.5}$$

Proof. From the smoothing properties of the transition semigroup associated to (2.2), combined with its controllability and the fact that $\|y\|^2$ is a Lyapunov function, one gets the existence of constants C and δ' such that

$$\|\mathcal{P}_t(x, y; \cdot) - \nu\|_{\mathrm{TV}} \leqslant C(1 + \|y\|^2)e^{-\delta' t}\,.$$

(See e.g. ref. 16 for further details.). Here $\|\mu - \nu\|_{\mathrm{TV}}$ denotes the total variation distance between the measures μ and ν. Cauchy-Schwarz furthermore yields the generic inequality

$$\Big|\int f\,d\mu - \int f\,d\nu\Big| \leqslant \sqrt{\|\mu - \nu\|_{\mathrm{TV}} \int f^2\,(d\mu + d\nu)} \tag{3.6}$$

The bound (3.5) immediately follows by combining Lemma 3.6 with (3.6). ∎

We also have a much stronger bound on the supremum in time of the solution:

Lemma 3.8. For every $\kappa > 0$ and every $T > 0$, there exist constants $\delta, C > 0$ such that

$$\mathbf{E} \sup_{t \in [0, T\varepsilon^{-2}]} \exp\big(\delta \|y(s)\|^2\big) \leqslant C\varepsilon^{-\kappa} e^{\delta \|y\|^2},$$

holds for every $\varepsilon \in [0, 1]$.

Proof. Let $\tilde{y}$ be the Ornstein-Uhlenbeck process defined by

$$\tilde{y}(t) = \frac{1}{\sqrt{\tau}} \int_0^t e^{-\frac{t-s}{\tau}} \sigma \, d\beta(s).$$

Then (see *e.g.* ref. 1), there exists constants c_1 and c_2 such that

$$\mathbf{P}\Big(\sup_{t \in [s, s+T]} \|\tilde{y}(t)\| > \lambda\Big) \leqslant c_1 e^{-c_2 \lambda^2},$$

for every $s > 0$. This immediately yields

$$\mathbf{P}\Big(\sup_{t \in [0, T\varepsilon^{-2}]} \|\tilde{y}(t)\| > \lambda\Big) \leqslant c_1 \varepsilon^{-2} e^{-c_2 \lambda^2},$$

which in turn implies that there exist constants c_3 and c_4 such that

$$\mathbf{E}\Big(\sup_{t \in [0, T\varepsilon^{-2}]} \exp\big(c_3 \|\tilde{y}(t)\|^2\big)\Big) \leqslant c_4 \varepsilon^{-2}.$$

The claim follows immediately by choosing $\delta = (c_3\kappa)/2$ and by noticing that there exists a constant c_4 such that $\|y(s)\| \leqslant \|\tilde{y}(s)\| + \|y\| + c_4$ for all $s > 0$ almost surely. ∎

4. PROOF OF THEOREM 2.1

Proof. By Theorem 3.3 we have $\Phi(y, z) \in \mathcal{C}^\infty(\mathbb{T}^n \times \mathbb{R}^n, \mathbb{R}^n)$, so we can apply the Itô formula to the function $\Phi\big(y_t^\varepsilon, x_t^\varepsilon\big)$ to obtain:

$$\begin{aligned}
\Phi\big(y_t^\varepsilon, x_t^\varepsilon\big) - \Phi(y, x) &= \frac{1}{\varepsilon^2} \int_0^t \mathcal{L}\Phi\big(y_s^\varepsilon, x_s^\varepsilon\big) \, ds + \frac{1}{\varepsilon} \frac{\sigma}{\sqrt{\tau}} \int_0^t \nabla_y \Phi\big(y_s^\varepsilon, x_s^\varepsilon\big) \, d\beta^\varepsilon(s) \\
&= -\frac{1}{\varepsilon^2} \frac{1}{\sqrt{\tau}} \int_0^t y_s^\varepsilon \, ds + \frac{1}{\varepsilon} \frac{\sigma}{\sqrt{\tau}} \int_0^t \nabla_y \Phi\big(y_s^\varepsilon, x_s^\varepsilon\big) \, d\beta^\varepsilon(s),
\end{aligned}$$

where we defined $\beta^\varepsilon(t)=\varepsilon\beta(\varepsilon^{-2}t)$ and we used (2.4) to get the second line. We also interpret $\nabla_y\Phi$ as a linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$. Thus we have:

$$\begin{aligned}\varepsilon x_t^\varepsilon &= \varepsilon x + \frac{1}{\varepsilon}\frac{1}{\sqrt{\tau}}\int_0^t y_s^\varepsilon\,ds \\ &= \varepsilon x - \varepsilon\big(\Phi\big(y_t^\varepsilon, x_t^\varepsilon\big) - \Phi(y,x)\big) + \frac{\sigma}{\sqrt{\tau}}\int_0^t \nabla_y\Phi\big(y_s^\varepsilon, x_s^\varepsilon\big)\,d\beta^\varepsilon(s) \\ &=: \varepsilon x + \varepsilon I_1^\varepsilon(t) + M^\varepsilon(t)\,. \end{aligned} \tag{4.1}$$

It follows from (3.4) and Lemma 3.8 that, for every $p>0$ there exists a constant C such that

$$\mathbf{E}\sup_{t\in[0,T]}\big|I_1^\varepsilon(t)\big|^p \leqslant C\varepsilon^{-\frac{p}{2}}\,.$$

It is therefore sufficient to show that (2.8) and (2.9) hold with μ_ε replaced by the law of the martingale term M^ε. We first show that (2.8) holds. This is equivalent to showing that, for every $\ell\in\mathcal{B}^*$ one can construct a random variable B_ℓ such that

$$\mathbf{E}|B_\ell - \ell(M^\varepsilon)|^p \leqslant C\varepsilon^{\alpha p}\,, \tag{4.2}$$

holds uniformly over $\|\ell\|\leqslant 1$, and such the law of B_ℓ is given by $\ell^*\mu$. We therefore fix $\ell\in\mathcal{B}^*$ with $\|\ell\|\leqslant 1$, which we interpret as an $\mathbb{R}^n$-valued measure with total mass (i.e. the sum of the masses of each of its components) smaller than 1. We also use the notation $\ell_t=\ell([t,T])$.

Integrating by parts, we can write

$$\ell(M^\varepsilon) = \int_0^T \langle M^\varepsilon(t), \ell(dt)\rangle = \frac{\sigma}{\sqrt{\tau}}\int_0^T \langle \ell(t), \nabla_y\Phi\big(y_t^\varepsilon, x_t^\varepsilon\big)\,d\beta^\varepsilon(t)\rangle\,.$$

We now define on the interval $[0,T]$ the $\mathbb{R}$-valued martingale M_ℓ^ε by

$$M_\ell^\varepsilon(t) = \frac{\sigma}{\sqrt{\tau}}\int_0^t \langle \ell(s), \nabla_y\Phi\big(y_s^\varepsilon, x_s^\varepsilon\big)\,d\beta^\varepsilon(s)\rangle\,.$$

According to the Dambis–Dubins–Schwartz theorem see ref. 24, Thm. 1.6, there exists a Brownian motion B such that $M_\ell^\varepsilon(t)$ can be written as

$$M_\ell^\varepsilon(t) = B\big(\langle M_\ell^\varepsilon, M_\ell^\varepsilon\rangle_t\big) = B\Big(\frac{\sigma^2}{\tau}\int_0^t \langle \ell(s), \big(\nabla_y\Phi\otimes\nabla_y\Phi\big)\big(y_s^\varepsilon, x_s^\varepsilon\big)\,\ell(s)\rangle\,ds\Big)\,.$$

On the other hand, the measure $\ell^*\mu$ is a centered Gaussian measure with variance $\int_0^T \langle \ell(s), \mathcal{K}^2 \ell(s)\rangle\, ds$, so we can choose B_ℓ to be given by

$$B_\ell = B_\ell^T\,, \qquad B_\ell^t = B\Big(\int_0^t \langle \ell(s), \mathcal{K}^2 \ell(s)\rangle\, ds\Big)\,.$$

We will actually show a stronger bound than (4.2), namely we will show that

$$J_\varepsilon^p := \mathbf{E} \sup_{t\in[0,T]} |B_\ell^t - M_\ell^\varepsilon(t)|^p \leqslant C\varepsilon^{\alpha p}\,. \tag{4.3}$$

We use the Hölder continuity of the Brownian motion B, together with the Cauchy–Schwarz inequality to derive the estimate

$$\begin{aligned} J_\varepsilon^p &\leqslant \mathbf{E}\Big(\mathrm{Höl}_\alpha^p(B) \sup_{0\leqslant t\leqslant T}\Big|\int_0^t \langle \ell(s), \Big(\frac{\sigma^2}{\tau}\big(\nabla_y\Phi\otimes\nabla_y\Phi\big)\big(y_s^\varepsilon, x_s^\varepsilon\big) - \mathcal{K}^2\Big)\ell(s)\rangle\, ds\Big|^{\alpha p}\Big) \\ &\leqslant \big(\mathbf{E}\,\mathrm{Höl}_\alpha^{2p}(B)\big)^{\frac12}\Big(\mathbf{E} \sup_{0\leqslant t\leqslant T}\Big|\int_0^t \langle \ell(s), H\big(y_s^\varepsilon, x_s^\varepsilon\big)\,\ell(s)\rangle\, ds\Big|^{2\alpha p}\Big)^{\frac12} \\ &\leqslant C\Big(\mathbf{E} \sup_{0\leqslant t\leqslant T}\Big|\int_0^t \langle \ell(s), H\big(y_s^\varepsilon, x_s^\varepsilon\big)\,\ell(s)\rangle\, ds\Big|^{2\alpha p}\Big)^{\frac12}\,, \end{aligned} \tag{4.4}$$

where we introduced the $n\times n$-matrix valued function

$$H(x,y) = \frac{\sigma^2}{\tau}\big(\nabla_y\Phi\otimes\nabla_y\Phi\big)(y,x) - \mathcal{K}^2\,.$$

In deriving the above estimate, we have used the fact that for $\alpha < \frac12$, the α-Hölder constant of a Brownian motion is uniformly bounded on every bounded interval (see ref. 24, Thm. 2.1).

Note now that since $\ell(t)$ is of bounded variation, $\ell(t)\otimes\ell(t)$ is also of bounded variation, so there exists a $n\times n$-matrix valued measure $\tilde\ell$ on $[0,T]$ such that $\ell(t)\otimes\ell(t) = \tilde\ell([t,T])$. Therefore, we can integrate by parts in (4.4) to obtain

$$\begin{aligned} J_\varepsilon^p &\leqslant C\Big(\mathbf{E} \sup_{0\leqslant t\leqslant T}\Big|\mathrm{Tr}\int_0^t\int_0^s H\big(y_r^\varepsilon, x_r^\varepsilon\big)\,dr\,\tilde\ell(ds)\Big|^{2\alpha p}\Big)^{\frac12} \\ &\leqslant C\Big(\mathbf{E} \sup_{0\leqslant t\leqslant T}\Big\|\int_0^t H\big(y_s^\varepsilon, x_s^\varepsilon\big)\,ds\Big\|^{2\alpha p}\Big)^{\frac12} \end{aligned}$$

Consider now the Poisson equation

$$-\mathcal{L}F = H\,. \tag{4.5}$$

By the definition of $\mathcal{K}^2$, we have $\int H(x,y)\,\nu(dx,dy)=0$ (for each component), and we furthermore have $\exp(-\delta\|y\|^2)H\in\mathcal{S}$ for every $\delta>0$. Therefore, using the same reasoning as in the proof of Theorem 3.3, equation (4.5) has a unique smooth solution satisfying

$$F(x,y)=e^{\frac{\delta}{2}\|y\|^2}\tilde{\mathcal{F}}(x,y)\,,\qquad \tilde{\mathcal{F}}\in\mathcal{S} \tag{4.6}$$

for every $\delta>0$. We can apply Itô formula to deduce as before that

$$\int_0^t H(y_s^\varepsilon,x_s^\varepsilon)\,ds=-\varepsilon^2\big(F(y_t^\varepsilon,x_t^\varepsilon)-F(y,x)\big)+\frac{\varepsilon}{\sqrt{\tau}}\int_0^t \nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\,\sigma\,d\beta(s)\,.$$

Therefore:

$$\begin{aligned}|J_\varepsilon^p|^2 \leqslant\ &\varepsilon^{4\alpha p}\mathbf{E}\sup_{t\in[0,T]}\|F(y_t^\varepsilon,x_t^\varepsilon)\|^{2\alpha p}\\ &+C\varepsilon^{2\alpha p}\mathbf{E}\sup_{t\in[0,T]}\Big\|\int_0^t\nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\,d\beta(s)\Big\|^{2\alpha p}.\end{aligned}$$

Combining Lemma 3.8 with (4.6), the first term can be bounded by

$$\varepsilon^{4\alpha p}\mathbf{E}\sup_{t\in[0,T]}\|F(y_t^\varepsilon,x_t^\varepsilon)\|^{2\alpha p}\leqslant C\varepsilon^{-2\alpha p}\,.$$

In order to control the second term, we use the Burkholder–Davis–Gundy inequality followed by Hölder's inequality, assuming that $p>\frac{1}{\alpha}$:

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,T]}\Big\|\int_0^t\nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\,d\beta(s)\Big\|^{2\alpha p}&\leqslant C\mathbf{E}\Big(\int_0^T\|\nabla_y F(y_s^\varepsilon,x_s^\varepsilon)\|^2\,ds\Big)^{\alpha p}\\ &\leqslant CT^{\alpha p-1}\sup_{t\in[0,T]}\mathbf{E}\|\nabla_y F(y_t^\varepsilon,x_t^\varepsilon)\|^{2\alpha p}\,.\end{aligned}$$

This is bounded independently of ε by (4.6) and Lemma 3.6, and so $J_\varepsilon^p\leqslant C\varepsilon^{\alpha p}$, for $p>\frac{1}{\alpha}$. When $p<\frac{1}{\alpha}$, one can bound J_ε^p using the higher order moments. This completes the proof of bound (4.2) and thus of the first part of Theorem 2.1.

The proof of the second part of Theorem 2.1 is obtained in a straightforward way as a particular case of (4.3) if one makes the choice $\ell=k\delta_T$.

We now turn to the proof of the third part of Theorem 2.1. For this, we start with some background material from refs. 12, 25. Let $P_p^M(\mathbb{R}^n)$

denote the set of all probability measures μ on $\mathbb{R}^n$ with finite pth moment $m_p(\mu)$ for some $p>2$ to be fixed later, and such that

$$m_p(\mu) \leqslant M\,.$$

Let $\rho(\mu_1, \mu_2)$ denote the Prokhorov metric; we introduce the metrics

$$d_1(\mu_1, \mu_2) = \sup_{\ell\in\mathbb{R}^n} \frac{|\chi_{\mu_1}(\ell) - \chi_{\mu_1}(\ell)|}{|\ell|}$$

and

$$\|\mu_1 - \mu_2\|_m^* = \sup\left\{\left|\int \phi(z) d(\mu_1(z) - \mu_2(z))\right|, \phi \in \mathcal{C}^\infty,\ \|\phi\|_m \leqslant 1\right\},$$

where $\|\cdot\|_m$ denotes the natural norm on $\mathcal{C}^m(\mathbb{R}^n)$. Now, a trivial modification of ref. 25, Thm. 2 gives

$$\|\mu_1 - \mu_2\|_{n+2}^* \leqslant C(M)\, d_1(\mu_1, \mu_2)^{\frac{2}{n+3}}\,,$$

where n is the dimension of the underlying space. Further, ref. 12, cor. 5.5 and ref. 25, Thm. 2 imply that

$$\rho(\mu_1, \mu_2) \leqslant c_m \left(\|\mu_1 - \mu_2\|_m^*\right)^{\frac{1}{m+1}}$$

for every $m>0$ and

$$|||\mu_1 - \mu_2|||_{2,W}^2 \leqslant C(M)\, \rho(\mu_1, \mu_2)^{\frac{p-2}{p}}\,.$$

Let μ_t^ε and μ_t denote the laws of $\varepsilon x_t^\varepsilon$ and the limiting Brownian motion at time t respectively, i.e. the images of μ_ε and μ under the map $x \mapsto x(t)$. With these notations, the considerations above yield

Lemma 4.1. Let the assumptions of Theorem 2.1 hold. Then, for every $\alpha \in (0, \frac{1}{2})$ and every $t \in [0, T]$, we have:

$$\left|\left|\left|\mu_t^\varepsilon - \mu_t\right|\right|\right|_{1,W} \leqslant C\varepsilon^{\frac{\alpha}{2(n+3)^2}}\,.$$

Here, the constant C depends only on $\mathbf{E}\exp\frac{1}{2}\|\sigma^{-1}y\|^2$.

Proof. From Theorem 2.1 and (2.7) we have $d_1(\mu_1, \mu_2) \leqslant \left|\!\left|\!\left| \mu_t^\varepsilon - \mu_t \right|\!\right|\!\right|_1 \leqslant C\varepsilon^\alpha$. Further, our bounds on the moments of the particle velocity imply that M is bounded independendly of ε for every $p > 0$. The parameter δ given by $1-\delta = \frac{p-2}{p}$, can thus be chosen arbitrarily small. Thus, for ε sufficiently small and $\delta > 0$, arbitrarily small we have:

$$\left|\!\left|\!\left| \mu^\varepsilon - \mu \right|\!\right|\!\right|_{2,W}^2 \leqslant C\rho(\mu^\varepsilon, \mu)^{1-\delta} \leqslant C \left(\|\mu^\varepsilon - \mu\|_{n+2} \right)^{\frac{1-\delta}{n+3}}$$
$$\leqslant C d_1(\mu^\varepsilon, \mu)^{\frac{1-\delta}{(n+3)^2}} \leqslant C\varepsilon^{\frac{\alpha(1-\delta)}{(n+3)^2}},$$

from which the estimate follows upon applying Cauchy–Schwarz inequality. The claim about the constant C is obtained by inspecting the bounds from Section 3. ▮

Fix now an integer $N > 0$, define $t_j = jT/N$, and define the map $\Pi_N : \mathcal{B} \to (\mathbb{R}^n)^N$ by $(\Pi_N x)_j = x(t_j)$. We first show that, for every $\gamma < \frac{1}{4(n+3)^2}$, there exists a constant C such that

$$\left|\!\left|\!\left| \Pi_N^* \mu_\varepsilon - \Pi_N^* \mu \right|\!\right|\!\right|_{1,W} \leqslant C\varepsilon^\gamma N^2, \tag{4.7}$$

for every $N > 0$. Lemma 4.1 indeed implies that if $\mathcal{P}_t^\varepsilon$ denotes the transition probabilities for the process $(\varepsilon x_t^\varepsilon, y_t^\varepsilon)$, π_2 denotes the projection on the first component, and $\mathcal{P}_t$ denotes the transition probabilities for a Brownian motion with covariance $\mathcal{K}$, one has for every $t \in [0, T]$

$$\left|\!\left|\!\left| \pi_2^* \mathcal{P}_t^\varepsilon(x, y; \cdot) - \mathcal{P}_t(x, \cdot) \right|\!\right|\!\right|_{1,W} \leqslant C(y)\, \varepsilon^\gamma,$$

where $C(y)$ is such that $\mathbf{E}C(y_t^\varepsilon)$ is bounded uniformily for $t \in [0, T]$. This implies that one can construct a Brownian motion B_t with covariance $\mathcal{K}$ such that

$$\mathbf{E}\|\varepsilon x_{t_j}^\varepsilon - B_{t_j}\| \leqslant C\varepsilon^\gamma + \mathbf{E}\|\varepsilon x_{t_{j-1}}^\varepsilon - B_{t_{j-1}}\|.$$

In particular, one has

$$\mathbf{E} \sup_j \|\varepsilon x_{t_j}^\varepsilon - B_{t_j}\| \leqslant \sum_{j=0}^N \mathbf{E}\|\varepsilon x_{t_j}^\varepsilon - B_{t_j}\| \leqslant C\varepsilon^\gamma N^2,$$

which implies (4.7) by definition.

Furthermore, the generalized Kolmogorov criteria (see ref. 24, Thm. 2.1) immediately implies that the α-Hölder constants of $\varepsilon x_t^\varepsilon$ and of the limiting Brownian motion B_t are bounded independently of ε for every $\alpha < 1/2$. Therefore,

$$\|\|\mu_\varepsilon - \mu\|\|_{1,W} \leqslant C\varepsilon^\gamma N^2 + \frac{C}{N^\alpha}.$$

Optimizing for N concludes the proof of Theorem 2.1. ∎

5. CONCLUSIONS

The problem of homogenization for periodic hypoelliptic diffusions was studied in this paper. It was proved that the rescaled particle position converges to a Brownian motion with a covariance matrix which can be computed in terms of the solution of the Poisson equation (2.4). Further, upper bounds on the convergence rate in several norms were obtained. Our analysis is purely probabilistic and this enables us to obtain more detailed information than what one could obtain from studying the problem at the level of the Kolmogorov equation. The convergence rate in the 1-Wasserstein metric, estimate (2.10), is almost certainly not sharp, it is however optimal in the sense that the p-Wasserstein metric is the strongest "natural" metric in which convergence is expected to hold, see Remark 2.8.

A very interesting question is whether a homogenization theorem of the form of Theorem 2.1 holds for random drifts $v(x,t)$ and, if yes, under what conditions on $v(x,t)$. We plan to come back to this issue in a future publication.

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